3. Ito, K. and Nisio, M., Stationary solutions of a stochastic differential equation. Matematika: Collected Translations and Surveys of Foreign Periodical Literature Vol. 11, №5, 1967.
4. Gikhman, I. I. and Skorokhod, A. V., Introduction of the Theory of Random Processes. Moscow, "Nauka", 1965.

Translated by A. Y.

THE TIME-OPTIMAL CONTROL PROBLEM IN SYSTEMS WITH CONTROLLING FORCES OF BOUNDED MAGNITUDE AND IMPULSE

PMM Vol. 34, №5, 1970, pp. 836-849
A. M. FORMAL'SKII
(Moscow)
(Received November 12, 1969)

The problem of time-optimal control is considered in the case where the controlling forces are bounded in magnitude and in impulse at the same time. The study is carried out with the aid of attainability domains. The case where the boundaries of these domains have plane portions and corners is considered. The problem of optimal control synthesis is solved for certain second-order systems with the indicated restrictions imposed on the controlling forces.

1. Statement of the problem. Let us consider the control system described by the following linear matrix differential equation with real constant coefficients:

$$dx/dt = Ax + Bu \tag{1.1}$$

Here $x = \| x_i \|$, $A = \| a_{ij} \|$, $B = \| b_{is} \|$, $u = \| u_s \|$ are matrices of order $(n \times 1)$, $(n \times n)$, $(n \times r)$, $(r \times 1)$, respectively. and $u_s = u_s(t)$ is a measurable function of time which satisfies the following restrictions simultaneously:

$$| u_s(t) | \leqslant M_s \qquad (M_s = \text{const} > 0) \tag{1.2}$$

$$\int_0^\infty | u_s(\tau) | d\tau \leqslant C_s^\circ \qquad (C_s^\circ = \text{const} > 0) \tag{1.3}$$

By b_s $(s = 1, \ldots, r)$ we denote the sth column of the matrix B ($b_s \neq 0$ for all $s = 1, \ldots, r$). Condition (1.2) expresses the boundedness of the controlling force, and condition (1.3) expresses (from the physical standpoint) the boundedness of the impulse of the controlling force. Inequality (1.3) in certain cases represents the limitation of the propellant capacity of a thruster.

We shall consider the problem of bringing system (1.1) to the origin in the minimum time by means of a control which satisfies conditions (1.2), (1.3) (e. g. see [1], and, among other things, the problem of synthesizing the time-optimal control.

When restriction (1.2) alone is imposed, the time-optimal control is, as we know [2-5], a relay control (we denote the minimum time in this case by $\theta = \theta(x)$). The problem of synthesizing such a control consists in splitting the space X_n composed of the phase coordinates $x_1, \ldots, x_n$ by the switching surfaces into domains in which the controls $u_s(t)$ assume the values M_s and $-M_s$ $(s = 1, \ldots, r)$. Once this splitting has been effected, the optimal control is known as a function of the phase coordinates

$u = u'(x)$.

The statement of the problem is altered when restrictions (1.2) and (1.3) are both imposed. In this case the optimal control at the initial instant depends not only on the initial-state vector x°, but also on the vector $C^\circ = \| C_s^\circ \|$, i.e. $u = u(x^\circ, C^\circ)$. At a present instant t we have $u = u(x, C)$, where x is the state of the system at the instant t, and where the vector $C = \| C_s \|$ characterizes the restriction on the impulse at this instant,

$$\int_t^\infty |u_s(\tau)| \, d\tau \leqslant C_s \qquad (s = 1, \ldots, r)$$

Let us introduce the coordinate x_{n+s} $(s = 1, \ldots, r)$ defined by the differential equation

$$\frac{dx_{n+s}}{dt} = -|u_s(t)|$$

The solution of this equation is of the form

$$x_{n+s}(t) = x_{n+s}(0) - \int_0^t |u_s(\tau)| \, d\tau$$

Let $x_{n+s}(0) = C_s^\circ$. Expression (1.3) then yields

$$\int_t^\infty |u_s(\tau)| \, d\tau \leqslant x_{n+s}(t)$$

i.e. the quantity $x_{n+s}(t)$ characterizes the impulse or the "propellant capacity" of the engine which produces the force u_s at the present instant t.

The problem of synthesizing the optimal control consists in constructing the function $u = u(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r})$. It is easy to see that in the domain

$$\theta(x_1, \ldots, x_n) \leqslant \frac{1}{M_s} x_{n+s} \qquad (s = 1, \ldots, r) \tag{1.4}$$

of the space X_{n+r} consisting of the phase coordinates $x_1, \ldots, x_n$ and the coordinates $x_{n+1}, \ldots, x_{n+r}$ the optimal control does not depend on the coordinates x_{n+s} $(s = 1, \ldots, r)$. Synthesis in this domain, i.e. in (1.4), can be effected by means of the function $u = u'(x)$.

2. Attainability domains. The solution of Eq. (1.1) is of the form

$$x(t) = e^{At}x^\circ + \int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau \tag{2.1}$$

Let $x(t) = 0$ for $t = T$. Equation (2.1) then yields

$$-x^\circ = \int_0^T e^{-A\tau} Bu(\tau) \, d\tau \tag{2.2}$$

Suppose that

$$\int_0^T |u_s(\tau)| \, d\tau \leqslant x_{n+s} = \text{const} \qquad (s = 1, \ldots, r) \tag{2.3}$$

We denote the set of measurable functions $u_s(t)$ which satisfy conditions (1.2) and (2.3) by $\Omega_s(T)$. The set of vector functions $u(t) = \| u_s(t) \|$ such that $u_s(t) \in \Omega_s(T)$ we denote by $\Omega(T)$. We also introduce the notation

$$v_s(T) = \int_0^T e^{-A\tau} b_s u_s(\tau) \, d\tau, \qquad v(T) = \sum_{s=1}^r v_s(T) = \int_0^T e^{-A\tau} Bu(\tau) \, d\tau \tag{2.4}$$

and consider the attainability domains

$$Q_s(T) = \{v_s(T): \quad u_s(t) \in \Omega_s(T)\}, \quad Q(T) = \sum_{s=1}^{r} Q_s(T) = \{v(T): u(t) \in \Omega(T)\}$$

in the space X_n.

Each of the sets $Q_s(T)$, and therefore the attainability domain $Q(T)$, has the following properties.

1°. Closure. 2°. Convexity. 3°. $Q_s(T)$ "grows" with growing T, i.e. $Q_s(T_1) \in Q_s(T_2)$ if $T_1 \leqslant T_2$. 4°. Symmetry with respect to the origin.

Property 1° follows from the fact that the set $Q_s(T)$ is a linear map of the set $\Omega_s(T)$, which is weakly compact in itself in the space $L_2[0, T]$. The latter can be proved by making use of the fact that a sphere is weakly compact in itself in the space $L_2[0, T]$ (see [6]). Properties 2°, 3°, 4° can be proved easily [3, 7, 8].

Let us take an arbitrary unit vector η $(1 \times n)$ and construct the support hyperplanes of the set $Q(T)$ orthogonal to the vector η. Properties 2° and 4° imply that there are two such planes and that they are symmetric to each other with respect to the origin (Fig. 1). The distance $d_\eta(T)$ from the origin to these planes is given by the expression [8, 9]

$$d_\eta(T) = \max_{v(T) \in Q(T)} (\eta v(T)) = \max_{u(t) \in \Omega(T)} \int_0^T \eta e^{-A\tau} Bu(\tau)\, d\tau \tag{2.5}$$

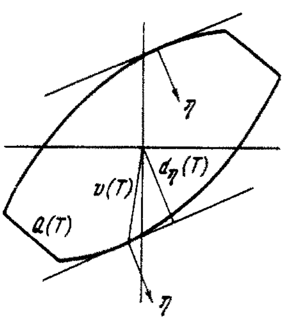

Fig. 1

Properties 1° and 2° imply that the set $Q(T)$ consists of those and only those points x whose coordinates satisfy the inequalities

$$|\eta x| \leqslant d_\eta(T) \tag{2.6}$$

for all possible unit vectors η.

The definition of the set $Q(T)$ implies that system (1.1) can be brought to the origin in the time T if and only if $x^\circ \in Q(T)$. By taking the limit as $T \to \infty$ in relations (2.5) and (2.6), we can find the controllability domain Q [8], i.e. the set of points of the space X_n from which the system can be brought to the origin by means of a control which satifies conditions (1.2), (1.3).

3. Determination of the distances $d_\eta(T)$. If the function $u(t)$ is subject to restrictions (1.2) (restrictions (2.3) do not apply), then the integral

$$J(u, \eta, T) = \sum_{s=1}^{r} J_s(u, \eta, T) = \sum_{s=1}^{r} \int_0^T \eta e^{-A\tau} b_s u_s(\tau)\, d\tau \tag{3.1}$$

attains its maximum under the control

$$u_s(t) = M_s \operatorname{sgn}[\eta e^{-At} b_s] \qquad (s = 1, \ldots, r) \tag{3.2}$$

If $M_s T \leqslant x_{n+s}$ $(s = 1, \ldots, r)$, then the control defined by (3.2) satisfies the relation $u(t) \in \Omega(T)$.

Now let us consider the problem of the maximum of the functional $J_s(u, \eta, T)$ under the assumption that

$$M_s T > x_{n+s} \tag{3.3}$$

We introduce the notation

$$E^{\sigma_s} = \{t \in [0, T]: |\eta e^{-At} b_s| \geqslant \sigma_s\} \quad (\sigma_s = \text{const}) \tag{3.4}$$

$$F^{\sigma_s} = \{t \in [0, T]: |\eta e^{-At} b_s| = \sigma_s\}$$

$$G^{\sigma_s} = \{t \in [0, T]: |\eta e^{-At} b_s| < \sigma_s\} \; (E^{\sigma_s} + G^{\sigma_s} = [0, T]) \tag{3.5}$$

$$u_s(t, \sigma_s) = \begin{cases} M_s \operatorname{sgn} [\eta e^{-At} b_s] & \text{for } t \in E^{\sigma_s} \\ 0 & \text{for } t \in G^{\sigma_s} \end{cases} \tag{3.6}$$

By $\mu\Phi$ we denote the Lebesgue measure [10] of the set $\Phi \in [0, T]$.

The function $\eta e^{-At} b_s$ is analytic. Hence, either $|\eta e^{-At} b_s| \equiv \sigma_s$ for some $\sigma_s \geqslant 0$ for all $t \in [0, T]$, so that $E^{\sigma_s} = F^{\sigma_s} = [0, T]$, $\mu E^{\sigma_s} = \mu F^{\sigma_s} = T$, or for all $\sigma_s \geqslant 0$ the equation $|\eta e^{-At} b_s| = \sigma_s$ is valid only for a finite number of points $t \in [0, T]$ so that $\mu F^{\sigma_s} = 0$.

First, suppose that $\mu F^{\sigma_s} = 0$ for all $\sigma_s \geqslant 0$. It is easy to show that in this case the quantity μE^{σ_s} varies continuously and monotonically from T to 0 as the quantity σ_s varies from the minimum value of the function $|\eta e^{-At} b_s|$ in the interval $[0, T]$ to its maximum value in the same interval. Condition (3.3) implies that there exists a unique σ_s° for which

$$\mu E^{\sigma_s^\circ} = \frac{1}{M_s} x_{n+s} \tag{3.7}$$

From (3.6) and (3.7) we see that $u_s(t, \sigma_s^\circ) \in \Omega_s(T)$. In [8, 11] arguments similar to those used to prove the von Neumann-Pearson lemma [3, 7] are adduced to show that the control $u_s(t, \sigma_s^\circ)$ (and only this control) maximizes the functional $J_s(u, \eta, T)$.

Let us assume now that $\mu F^{\sigma_s} = T$ for some $\sigma_s \geqslant 0$, i.e. that

$$\eta e^{-At} b_s \equiv D_s = \text{const} \qquad (\sigma_s = |D_s|)$$

In this case all the controls from the set $\omega_s(\eta, T)$ of functions satisfying inequality (1.2) and the conditions

$$[D_s u_s(t)] \geqslant 0, \qquad \int_0^T u_s(\tau)\, d\tau = x_{n+s} \operatorname{sgn} D_s$$

are maximizing controls.

We note that expression (3.2) follows from (3.6) for $\sigma_s = 0$. For this reason we can assume that for $M_s T \leqslant x_{n+s}$ the maximizing control is $u_s(t, \sigma_s^\circ)$, where $\sigma_s^\circ = 0$ $(E^{\sigma_s^\circ} = [0, T])$.

We obtain the following expression for the distance $d_\eta(T)$:

$$d_\eta(T) = \sum_{s=1} M_s \int_{E^{\sigma_s^\circ}} |\eta e^{-A\tau} b_s|\, d\tau \tag{3.8}$$

It can be shown that the distance $d_\eta(T)$ is a continuous function of the vector η and of the quantity T. From this and from inequalities (2.6) we conclude that the sets $Q_s(T)$ and $Q(T)$ have the following "continuity" property.

5°. If the point v is an interior point of the set $Q_s(T)$, then there exists a $T_1 < T$, such that $v \in Q_s(T_1)$.

We note that Property 5° can be proved by a method quite similar to that used in [5] (p. 88) to prove the analogous property under restrictions of the type (1.2) only.

4. Plane portions of the boundaries of the attainability domains $Q_s(T)$ and $Q(T)$. Substituting into (2.4) the controls $u_s(t)$ which maximize the functionals $J_s(u, \eta, T)$ $(s = 1, \ldots, r)$, we obtain the vector $v(T)$ of the coordinates of the point which is common to one of the two support hyperplanes and the set $Q(T)$ (the point of tangency). Here $v_s(T)$ is the vector of the coordinates of the point of tangency of the set $Q_s(T)$ with one of the support hyperplanes of the set $Q_s(T)$ orthogonal to the vector η.

The set $Q(T)$ has a unique point of tangency if and only if all the sets $Q_s(T)$ $(s = 1, \ldots, r)$ have a unique point of tangency. If the equation which maximizes the functional $J_s(u, \eta, T)$ is uniquely defined, then the set $Q_s(T)$ has a unique point of tangency. The maximizing control $u_s(t)$ is defined ambiguously in the two cases

$$\eta e^{-At} b_s \equiv D_s = \text{const} \neq 0, \qquad M_s T > x_{n+s} \tag{4.1}$$

$$\eta e^{-At} b_s \equiv 0 \tag{4.2}$$

In the simplest case (4.2) the functional $J_s(u, \eta, T) = 0$ for any controls $u_s(t)$. In this case the set $Q_s(T)$ belongs entirely to the plane $\eta x = 0$. A vector η for which condition (4.2) holds exists if and only if [4, 12] the rank ρ_s of the matrix $W_s = \| b_s, Ab_s, \ldots, A^{n-1} b_s \|$ is smaller than n. The set $Q_s(T)$ belongs to the subspace X_{ρ_s} of dimension ρ_s. The same situation obtains when restrictions (1.2) alone are imposed.

In case (4.1) all the controls $u_s(t) \in \omega_s(\eta, T)$ are maximizing controls. The set $P_s(\eta, T)$ of points of tangency is defined by the expression

$$P_s(\eta, T) = \{v_s(T): \quad u_s(t) \in \omega_s(\eta, T)\}$$

This set belongs to some hyperplane $\Pi_s(\eta)$ orthogonal to the vector η.

The set $P_s(\eta, T)$ has Properties 1°, 2°, 3°. It is easy to show that the set $P_s(\eta, T)$ is $(\rho_s - 1)$-dimensional. For example, if $\rho_s = n$, then the n-dimensional boundary of the set $Q_s(T)$ contains $(n - 1)$-dimensional plane portions, i.e. the set $Q_s(T)$ is not strictly convex. This situation cannot exist if restrictions (1.2) alone are imposed, consequently the set $Q_s(T)$ is always strictly convex [5].

The set $P_s(\eta, T)$ has Property 5°. In other words, if v is an interior point of the set $P_s(\eta, T)$ in the plane $\Pi_s(\eta)$, then there exists a $T_1 < T$ such that $v \in P_s(\eta, T_1)$.

Let us take an arbitrary point belonging to the boundary in the plane $\Pi_s(\eta)$ of the set $P_s(\eta, T)$ and construct a support hyperplane of the set $P_s(\eta, T)$ at this point. There is an infinite number of such planes in the space X_{ρ_s}. Among them is a plane π such that the vector η' orthogonal to it varies arbitrarily little from the vector η but does not coincide with the latter. It is possible to ensure that

$$\operatorname{sgn}[\eta' e^{-At} b_s] = \operatorname{sgn}[\eta e^{-At} b_s] = \operatorname{sgn} D_s \qquad \text{for } t \in [0, T]$$

Hence, for vectors η' sufficiently close to the vector η the controls $u_s(t) \in \Omega_s(T)$ which maximize the functional $J_s(u, \eta, T)$ also belong to the set $\omega_s(\eta, T)$. The points $v_s(T)$ resulting from these controls belong to the set $P_s(\eta, T)$.

The support hyperplane of the set $Q_s(T)$ which is orthogonal to the vector η' cannot lie closer to the origin than the plane π. On the other hand, this support hyperplane cannot lie further than the plane π from the origin, since it contains points from the set $P_s(\eta, T)$. Hence, this support hyperplane coincides with the plane π.

We have thus proved that a nonunique support hyperplane of the set $Q_s(T)$ exists at the boundary points of the set $P_s(\eta, T)$; in other words, the boundary points of the set $P_s(\eta, T)$ are "corner points" for the boundary of the domain $Q_s(T)$.

The dimension of the set $Q(T)$ is equal to the rank R of the matrix $W = \| W_1, \ldots$

$\ldots, W_r \|$, i.e. the set $Q(T)$ lies in some subspace X_R of dimension R.

Suppose that for some vector $\eta \in X_R$ conditions (4.1) hold for $s = 1, \ldots, r_1$ and conditions (4.2) for $s = r_1 + 1, \ldots, r$. We introduce the notation

$$P(\eta, T) = \sum_{s=1}^{r} P_s(\eta, T), \qquad P_s(\eta, T) = Q_s(T) \qquad (s = r_1 + 1, \ldots, r)$$

The distance $d_\eta(T)$ does not depend on T; we denote this distance by d_η ($d_\eta \neq 0$). The set $P(\eta, T)$ lies in the plane $\Pi(\eta)$ ($\eta x = d_\eta$). Its dimension does not exceed $R - 1$. The set $P(\eta, T)$, as the set $P_s(\eta, T)$, possesses Properties 1°, 2°, 3°, 5°. The boundary points of the set $P(\eta, T)$ are corner points for the boundary of the domain $Q(T)$.

5. The optimal control. Let the initial state $x^\circ \in Q \in X_R$. We denote by T° the minimum value of T for which $x^\circ \in Q(T)$. In other words, T° is the minimum value of T for which there exists a control $u(t) \in \Omega(T)$ such that relation (2.2) holds. Property 5° of the set $Q(T)$ implies that the point x° belongs to the boundary of the set $Q(T^\circ)$. Let us construct in the space X_R the support hyperplane of the set $Q(T^\circ)$ which passes through the point x°. This plane splits X_R into two half-spaces. Let $\eta^{(1)} \in X_R$ be the vector orthogonal to this plane and directed into the half-space where the set $Q(T^\circ)$ is situated. The time-optimal control clearly maximizes the integral $J(u, \eta^{(1)}, T^\circ)$.

Suppose that for all $s = 1, \ldots, r$ the control which maximizes the functional $J_s(u, \eta^{(1)}, T^\circ)$ is defined ambiguously, i.e. that

$$\eta^{(1)} e^{-At} b_s \equiv D_s^{(1)} = \text{const} \neq 0, \qquad M_s T^\circ > x_{n+s} \qquad (s = 1, \ldots, r_1^{(1)})$$

$$\eta^{(1)} e^{-At} b_s \equiv 0 \qquad (s = r_1^{(1)} + 1, \ldots, r)$$

Then $x^\circ \in P(\eta^{(1)}, T^\circ) \in \Pi(\eta^{(1)})$. Property 5° of the set $P(\eta^{(1)}, T)$ implies that the point x° belongs to the boundary of the set $P(\eta^{(1)}, T)$, i.e. that it is a corner point of the boundary of the set $Q(T^\circ)$. This means that at the point x° we can construct another support hyperplane of the set $Q(T^\circ)$ with the orthogonal vector $\eta^{(2)} \in \in X_R$ ($\eta^{(2)} \neq \eta^{(1)}$). If the maximizing controls for all s are defined ambiguously for the vector $\eta^{(2)}$ as well, then $x^\circ \in P(\eta^{(2)}, T^\circ) \in \Pi(\eta^{(2)})$, where $\Pi(\eta^{(2)})$ is a hyperplane which does not coincide with $\Pi(\eta^{(1)})$. This means, in turn, that the point x° belongs to the set $P(\eta^{(1)}, T^\circ) \times P(\eta^{(2)}, T^\circ)$ of dimension not higher than $R - 2$; moreover, it belongs to the boundary of this set. This makes it possible to construct at x° another support hyperplane of the set $Q(T^\circ)$ with an orthogonal vector $\eta^{(3)} \in X_R$ (the vectors $\eta^{(1)}, \eta^{(2)}, \eta^{(3)}$ are linearly independent). Suppose that by continuing this process we have succeeded in constructing the linearly independent vectors $\eta^{(k)} \in X_R$ ($k = 1, \ldots, R$) for each of which

$$\eta^{(k)} e^{-At} b_s \equiv D_s^{(k)} = \text{const} \neq 0, \quad M_s T^\circ > x_{n+s} \quad (s = 1, \ldots, r_1^{(k)})$$

$$\eta^{(k)} e^{-At} b_s \equiv 0 \qquad (s = r_1^{(k)} + 1, \ldots, r)$$

Since $b_s \neq 0$ ($s = 1, \ldots, r$), it follows that for every s there exists a k for which $\eta^{(k)} e^{-At} b_s \not\equiv 0$. Then $M_s T^\circ > x_{n+s}$ for $s = 1, \ldots, r$; in other words,

$$T^\circ > N = \max_{1 \leq s \leq r} (x_{n+s} / M_s)$$

The set $Q_s(T)$ $(s = 1, \ldots, r)$ is a segment which "grows" as T varies from 0 to x_{n+s} / M_s, and remains constant for $T \geqslant x_{n+s} / M_s$. The set $Q(T)$ is a polyhedron which does not depend on T for $T \geqslant N$. From this we infer that the optimal time $T^\circ \leqslant N$, which contradicts what we said previously. Hence, in constructing the vectors $\eta^{(k)}$ we encounter a $k \leqslant R$ such that the maximizing control remains uniquely defined for at least one value of the index s. Let this control $u_s^\circ(t)$ (which is optimal) be defined uniquely for $s = 1, \ldots, r_1$. As is evident from Sect. 3 of the present paper, the control $u_s^\circ(t) = u_s(t, \sigma_s^\circ)$ $(s = 1, \ldots, r_1)$ is defined by relations (3.4)–(3.6) in which $T = T^\circ$.

If $r_1 = r$, then the optimal control is defined completely. Let us suppose that $r_1 < r$ and find the controls $u_s^\circ(t)$ $(s = r_1 + 1, \ldots, r)$ for which Eq. (2.2) holds when $T = T^\circ$. To this end we substitute the resulting functions $u_s^\circ(t)$ $(s = 1, \ldots, r_1)$ into (2.2) and replace (2.2) by the relation

$$-\xi^\circ = \sum_{s=r_1+1}^{r} \int_0^T e^{-A\tau} b_s u_s(\tau)\, d\tau \qquad \left(\xi^\circ = x^\circ + \sum_{s=1}^{r_1} \xi_s, \quad \xi_s = \int_0^{T^\circ} e^{-A\tau} b_s u_s^\circ(\tau)\, d\tau\right) \quad (5.1)$$

The set

$$Q^{r_1}(T) = \sum_{s=r_1+1}^{r} Q_s(T)$$

has Properties 1°–5°. We denote by T^1 the minimum value of T for which $\xi^\circ \in Q^{r_1}(T)$; clearly, $T^1 \leqslant T^\circ$. The point ξ° belongs to the boundary of the set $Q^{r_1}(T^1)$. As above, we can prove that there exists a support hyperplane of this set which passes through the point ξ° with a vector η such that the maximizing control is defined uniquely for at least one of the sequence $r_1 + 1, \ldots, r$ of values of the index s. If the maximizing control can be defined uniquely for all $s = r_1 + 1, \ldots, r$ then the controls which realize Eqs. (5.1) for $T = T^\circ$ can be taken in the form

$$u_s^\circ(t) = \begin{cases} u_s(t, \sigma_s^\circ) & \text{for } 0 \leqslant t \leqslant T^1 \\ 0 & \text{for } T^1 < t \leqslant T^\circ \end{cases} \quad (5.2)$$

Here the function $u_s(t, \sigma_s^\circ)$ $(s = r_1 + 1, \ldots, r)$ is defined by relations (3.4)–(3.6) in which $T = T^1$.

For $T^1 < T^\circ$ the controls $u_s(t)$ $(s = r_1 + 1, \ldots, r)$ which realize Eq. (5.1) for $T = T^\circ$ are not unique and can be defined not only by means of formula (5.2).

If the maximizing control is uniquely defined only for $s = r_1 + 1, \ldots, r_2$ $(r_2 < r)$, then to determine the controls $u_s^\circ(t)$ $(s = r_2 + 1, \ldots, r)$ we must substitute controls (5.2) for $s = r_1 + 1, \ldots, r_2$ into (5.1) and develop an argument similar to that above. Proceeding in this manner, we can find all the controls $u_s^\circ(t)$ $(s = 1, \ldots, r)$ which realize Eq. (2.2) for $T = T^\circ$, and thereby determine the optimal control for the state x°.

Thus the optimal control $u_s^\circ(t)$ assumes the values $-M_s$, 0, M_s. The problem of synthesis consists in splitting the space X_{n+r} by the switching surfaces into domains in which the control assumes the appropriate values.

6. Second-order systems (*). Consider the system

---

*) The following students participated in developing the expressions appearing in this section: A. Ershov, V. Trofimov, S. Naumov, N. Gorushkina and V. Karandeev.

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = \lambda x_2 + \nu x_1 + u \tag{6.1}$$

Here $r = 1$, so we omit the index s. Without limiting generality we can assume that $M = 1$.

First let $\lambda = \nu = 0$; then we have

$$\eta e^{-At} b = \sin\varphi - t\cos\varphi \qquad (\cos\varphi = \eta_1, \ \sin\varphi = \eta_2) \tag{6.2}$$

Let us construct the attainability domain $Q(T)$ for $T > x_3$.

Let $\varphi \neq {}^1/_2\,\pi$. Carrying out some elementary operations, we find that the set E^{σ° consists of the segments

$$\begin{array}{ll} [T - x_3, \ T] & \text{for } \operatorname{tg}\varphi \leqslant {}^1/_2\,(T - x_3) \\ [0, \ \operatorname{tg}\varphi - {}^1/_2\,(T - x_3)], \ [\operatorname{tg}\varphi + {}^1/_2\,(T - x_3), \ T] & \text{for } {}^1/_2\,(T - x_3) < \operatorname{tg}\varphi < {}^1/_2\,(T + x_3) \\ [0, \ x_3] & \text{for } {}^1/_2\,(T + x_3) \leqslant \operatorname{tg}\varphi \end{array}$$

Hence we find that

$$\text{if} \quad -{}^1/_2\pi < \varphi \leqslant \operatorname{arctg}\,[{}^1/_2\,(T - x_3)], \tag{6.3}$$

$$u(t, \sigma^\circ) = \begin{cases} 0 & \text{for } 0 \leqslant t < T - x_3 \\ -1 & \text{for } T - x_3 \leqslant t \leqslant T \end{cases} \tag{6.4}$$

$$\text{if} \quad \operatorname{arctg}\,[{}^1/_2\,(T - x_3)] < \varphi < \operatorname{arctg}\,[{}^1/_2\,(T + x_3)], \tag{6.5}$$

$$u(t, \sigma^\circ) = \begin{cases} 1 & \text{for } 0 \leqslant t < \operatorname{tg}\varphi - {}^1/_2\,(T - x_3) \\ 0 & \text{for } \operatorname{tg}\varphi - {}^1/_2\,(T - x_3) \leqslant t < \operatorname{tg}\varphi + {}^1/_2\,(T - x_3) \\ -1 & \text{for } \operatorname{tg}\varphi + {}^1/_2\,(T - x_3) \leqslant t \leqslant T \end{cases} \tag{6.6}$$

$$\text{if} \quad \operatorname{arctg}\,[{}^1/_2\,(T + x_3)] < \varphi < {}^1/_2\pi, \tag{6.7}$$

$$u(t, \sigma^\circ) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant x_3 \\ 0 & \text{for } x_3 < t \leqslant T \end{cases} \tag{6.8}$$

For ${}^1/_2\,\pi < \varphi < {}^3/_2\pi$ the control $u(t, \sigma^\circ)$ can be obtained by multiplying functions (6.4), (6.6), (6.8) by -1. Thus, by virtue of the monotonousness of function (6.2) it turns out that the control $u(t, \sigma^\circ)$ assumes on $[0, T]$ each of the three values $-1, 0, 1$ not more than once.

For the distance $d_\eta(T)$ we obtain the expression

$$d_\eta(T) = \begin{cases} {}^1/_2 x_3 (2T - x_3)\cos\varphi - x_3\sin\varphi & \text{under condition (6.3)} \\ (\operatorname{tg}\varphi - T)\sin\varphi + {}^1/_2\,[T^2 - {}^1/_2\,(T - x_3)^2]\cos\varphi & \text{under condition (6.5)} \\ x_3\sin\varphi - {}^1/_2 x_3^2\cos\varphi & \text{under condition (6.7)} \\ x_3 & \text{for } \ \varphi = \pm{}^1/_2\pi \end{cases} \tag{6.9}$$

The boundary of the domain $Q(T)$ is the envelope of the straight support lines. This envelope can be readily determined from expression (6.9). It turns out that the domain $Q(T)$ is bounded by the two straight lines

$$x_2 = \pm x_3 \tag{6.10}$$

and by the two parabolas

$$x_1 = \pm\,[{}^1/_2\,(x_2 \mp T)]^2 \mp {}^1/_2\,[T^2 - {}^1/_2\,(T - x_3)^2]$$

Figure 2 shows the domains $Q(T)$ at $T = 1.0, \ 1.5, \ 2.0$ for $x_3 = 1$. At $T > x_3$ the boundaries of these domains have two plane portions and four corner points. The controllability domain Q which results from $Q(T)$ as $T \to \infty$ is the set of points lying between straight lines (6.10). Moreover, the domain Q includes the straight-line segments

$$x_1 \leqslant -{}^1/_2\, x_3^2,\quad x_2 = x_3$$

$$x_1 \geqslant {}^1/_2\, x_3^2,\quad x_2 = -x_3 \qquad (6.11)$$

(the thick segments in Fig. 2).

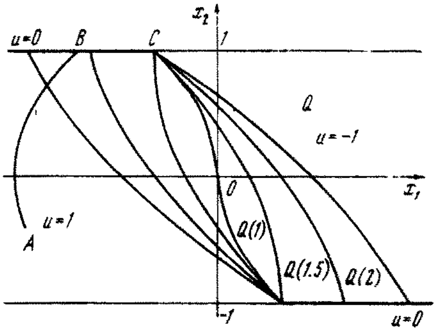

Fig. 2

For points belonging to Q (T) for $T \leqslant x_3$, the optimal control is a pure relay control and the switching line is the curve [5]

$$x_1 = -{}^1/_2\, x_2 \mid x_2 \mid \qquad (6.12)$$

Making use of expressions (6.3)–(6.8), we can find the value u (t, σ°) of the optimal control for $t = 0$, which enables us to solve the synthesis problem. It turns out that u $(0, \sigma^\circ) = 1$ on the set of points satisfying the conditions

$$x_1 < -{}^1/_2\, x_2 \mid x_2 \mid, \quad \mid x_2 \mid < x_3 \qquad (6.13)$$

or the conditions (6.14)

$$x_1 = -{}^1/_2\, x_2 \mid x_2 \mid, \qquad -x_3 \leqslant x_2 < 0$$

On the set of points satisfying the conditions

$$x_1 < -{}^1/_2\, x_3^2, \qquad x_2 = x_3 \qquad (6.15)$$

we have u $(0, \sigma^\circ) = 0$. Since the phase portrait of an optimal system is symmetric with respect to the origin, we can readily find u $(0, \sigma^\circ)$ at the points symmetric to the points of set (6.13)–(6.15). Figure 2 shows one of the possible optimal trajectories (curve ABCO).

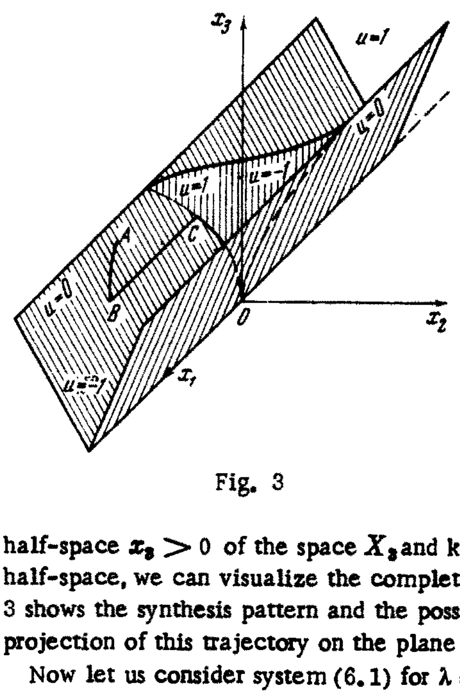

Fig. 3

Considering relations (6.10)–(6.15) in the half-space $x_3 > 0$ of the space X_3 and knowing the value of u (0) at each point of this half-space, we can visualize the complete pattern of optimal control synthesis. Figure 3 shows the synthesis pattern and the possible optimal trajectory ABCO (Fig. 2 shows the projection of this trajectory on the plane $x_3 = 0$).

Now let us consider system (6.1) for $\lambda \neq 0$, $\nu = 0$. In canonical variables (for which we retain the symbols x_1 and x_2) this system assumes the form

$$\dot{x}_1 = \lambda x_1 + x_2, \qquad \dot{x}_2 = u$$

Omitting the details presented in our analysis of the case $\lambda = 0$, we shall merely describe the results.

As in the case $\lambda = 0$, the domains Q (T) for $T > x_3$ have two plane portions and four corner points. For $\lambda < 0$ the controllability domain Q consists of points satisfying the condition

$$\mid x_2 \mid < x_3 \qquad (6.16)$$

and also of points belonging to certain portions of the boundary of set (6.16). For $\lambda > 0$ the domain Q is also bounded by the curves

$$x_1 = -\lambda^{-1}x_2 \pm \lambda^{-2}\{1 - \exp[-{}^1/_2\lambda(x_3 \pm x_2)]\} \tag{6.17}$$

The points of these curves do not belong to Q. We shall not derive (6.17) because of lack of space.

The pattern of optimal control synthesis in the half-space $x_3 > 0$ of the space X_3 for $\lambda \neq 0$ is qualitatively similar to the pattern for $\lambda = 0$. The role of surface (6.12) in this case is played by the surface [5]

$$x_1 = -\lambda^{-1}x_2 + \lambda^{-2}[1 - \exp(-\lambda|x_2|)]\operatorname{sgn}x_2 \tag{6.18}$$

The optimal control $u(x_1, x_2, x_3) = 0$ at the boundary points of the set (6.16) which belong to the domain Q but do not belong to surface (6.18). The optimal control $u(x_1, x_2, x_3) = 1$ at points of domain (6.16) lying to one side of surface (6.18) and on that part of surface (6.18) which belongs to Q. At the remaining points of the domain Q we have $u(x_1, x_2, x_3) = -1$.

Let us consider system (6.1) for $\lambda = 0$, $\nu < 0$. Without limiting generality we can assume that $\nu = -1$, whereupon system (6.1) becomes

$$x_1^{\cdot} = x_2, \qquad x_2^{\cdot} = -x_1 + u \tag{6.19}$$

The function $\eta e^{-At}b$ is of the form

$$\eta e^{-At}b = \sin(\varphi - t) \tag{6.20}$$

In contrast to the preceding examples, the identity $|\eta e^{-At}b| \equiv \sigma$ does not hold for any φ regardless of the value of $\sigma = \text{const}$. Hence, (6.20) enables us to define the control $u(t, \sigma^\circ)$ uniquely for any φ,

$$u(t, \sigma^\circ) = \begin{cases} \operatorname{sgn}[\sin(\varphi - t)] & \text{for } |\sin(\varphi - t)| \geqslant \sigma^\circ \\ 0 & \text{for } |\sin(\varphi - t)| < \sigma^\circ \end{cases} \tag{6.21}$$

and the boundary of the domain $Q(T)$ has no plane portions.

The problem of minimizing integral (1.3) for $|u| \leqslant 1$ is solved in [13] for $\lambda = \nu = 0$ and in [14] for $\lambda = 0$, $\nu = -1$. The optimal control in this problem has the same structure as control (6.4), (6.6) and (6.21). Let

$$k_1 = \left[\frac{x_3}{\pi - 2\arcsin\sigma^\circ}\right]$$

(k_1 is the whole part of the expression in square brackets), k_2 is the number of zeros of the equation $|\sin\delta| = \sigma^\circ$, where $\delta = \varphi - t$ in the interval $[\varphi - T, \varphi]$. Let us suppose that $|\varphi| < {}^1/_2\pi$; then σ° satisfies one of the following pair of relations:

$$\begin{aligned} 2k_1 &= k_2, & x_3 &= (\pi - 2\arcsin\sigma^\circ)k_1 \\ 2k_1 &= k_2, & x_3 &= T - 2k_1\arcsin\sigma^\circ \end{aligned} \tag{6.22}$$

$$\begin{aligned} 2k_1 &= k_2 - 1, & x_3 &= \pi k_1 - (2k_1 + 1)\arcsin\sigma^\circ + \varphi \\ 2k_1 &= k_2 - 1, & x_3 &= T - \varphi - (2k_1 + 1)\arcsin\sigma^\circ \\ 2k_1 &= k_2 - 2, & x_3 &= T - 2(k_1 + 1)\arcsin\sigma^\circ \end{aligned} \tag{6.23}$$

These relations can be readily established by considering Fig. 4, which shows the graph of the function $|\sin\delta|$. This graph represents the case corresponding to condition (6.23). In this case the set E^{σ° consists of k_1 segments of length $\pi - 2\arcsin\sigma^\circ$ and of one segment of length $T - \varphi - \pi k_1 - \arcsin\sigma^\circ$ (the set E^{σ° is indicated by the thick lines in Fig. 4).

Let us find the controllability domain Q. As $T \to \infty$ we have $\sigma^\circ \to 1$, $k_1 \to \infty$. Let

$T - \varphi = \pi k_1 + {}^1/_2\,\pi$; then (as we see from Fig. 4) for each fixed φ for sufficiently large T the quantity σ° satisfies relations (6.23). After some simple operations we arrive at the following expression for such values of T :

$$d_\eta(T) = \int_0^T \sin(\varphi - \tau)\, u(\tau, \sigma^\circ)\, d\tau = (2k_1 + 1) \sin \frac{x_3}{2k_1 + 1}$$

This implies that

$$\lim_{T\to\infty} d_\eta(T) = \lim_{k_1\to\infty} (2k_1 + 1) \sin \frac{x_3}{2k_1 + 1} = x_3$$

For $|\varphi| = {}^1/_2\pi$ we obtain the same result.

Thus, the controllability domain Q is the interior of a disk of radius x_3; in other words, it is described by the inequality

$$x_1^2 + x_2^2 < x_3^2 \qquad (6.24)$$

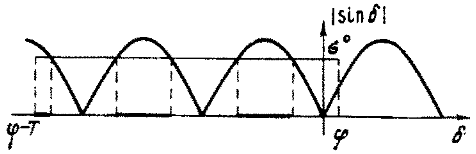

Fig. 4

The fact that the domain Q can be nothing other than a disk also follows from the fact that the phase trajectories of system (6.19) for $u(t) \equiv 0$ are circles. In the half-space $x_3 > 0$ of the space X_3 domain (6.24) is the interior of a cone.

Now let us consider the synthesis problem.

To this end we find on the plane X_2 the set D_0 of the points x° at which the optimal control at the initial instant is equal to zero. Let us suppose that $|\varphi| < {}^1/_2\,\pi$. Consideration of Fig. 4 then shows that $u(0, \sigma^\circ) = 0$ for those and only those values of T and φ for which $|\varphi| < \arcsin\sigma^\circ$. This inequality is valid only under condition (6.22) or (6.23). To determine the set D_0 we must substitute function (6.21) into (2.2) with allowance for conditions (6.22) and (6.23). Making use of the symmetry of the phase portrait of an optimal system, we find that the set of points $x \in D_0$ satisfies the relations

$$x_1 = \pm \int_0^T \sin\tau\, u(\tau, \sigma^\circ)\, d\tau, \qquad x_2 = \mp \int_0^T \cos\tau\, u(\tau, \sigma^\circ)\, d\tau \qquad (6.25)$$

Under condition (6.22) the control $u(t, \sigma^\circ) = 0$ for

$$\pi k_1 - \arcsin\sigma^\circ + \varphi < t < \pi k_1 + \arcsin\sigma^\circ + \varphi$$

Hence, as we see from (6.25), this condition need not be considered. We therefore assume that T and σ° in (6.25) satisfy condition (6.23). The set D_0 is biparametric: one parameter is the quantity φ satisfying the inequality $|\varphi| < \arcsin\sigma^\circ$; the other parameter is either T or σ°. Substituting the relations $\arcsin\sigma^\circ = \pm\varphi$ into (6.25), we obtain the boundaries of the set D_0.

Setting $\arcsin\sigma^\circ = \varphi$ and carrying out certain appropriate operations, we obtain the parametric equations of one of the boundaries of the set D_0,

$$x_1 = \mp\{(k_1 + 1)\cos 2\varphi + k_1 + (-1)^{k_1+1}\cos[x_3 + 2(k_1 + 1)\varphi]\} \quad k_1 = \left[\frac{x_3}{\pi - 2|\varphi|}\right]$$

$$x_2 = \mp\{(k_1 + 1)\sin 2\varphi + (-1)^{k_1+1}\sin[x_3 + 2(k_1 + 1)\varphi]\} \quad \varphi \in [0, {}^1/_2\pi) \quad (6.26)$$

Setting $\arcsin\sigma^\circ = -\varphi$, we obtain the equations of the other boundary of the set D_0,

$$x_1 = \mp\{k_1 \cos 2\varphi + k_1 + 1 + (-1)^{k_1+1} \cos(x_3 - 2k_1\varphi)\} \qquad (6.27)$$

$$x_2 = \mp\{k_1 \sin 2\varphi + (-1)^{k_1+1} \sin(x_3 - 2k_1\varphi)\} \qquad \varphi \in (-{}^1/_2\,\pi, 0]$$

The portions of curves (6.26),(6.27) are smooth in those ranges of φ-values in which k_1 remains constant.

As $\varphi \to {}^1/_2\pi$ expressions (6.26) yield $x_2 \to \mp x_3$, $x_1 \to 0$; as $\varphi \to -{}^1/_2\pi$ expressions (6.27) yield $x_2 \to \pm x_3$, $x_1 \to 0$. For $\varphi = 0$ Eqs. (6.26) and (6.27) assume the same form

$$x_1 = \mp[2k_1 + 1 + (-1)^{k_1+1} \cos x_3], \qquad x_2 = \mp[(-1)^{k_1+1} \sin x_3] \qquad (6.28)$$

Here

$$k_1 = \left[\frac{x_3}{\pi}\right]$$

Setting $x_3 = \pi k_1 + \alpha$, where $0 \leqslant \alpha < \pi$, we find that

$$x_1 = \mp[2k_1 + 1 - \cos\alpha], \qquad x_2 = \pm \sin\alpha \qquad (6.29)$$

Considering x_3 as a parameter, we find that curve (6.28) or (6.29) constitutes the switching line L (see [5]) for system (6.19) under the condition $|u| \leqslant 1$ alone. Consequently, for $x_3 = \text{const}$ all four curves (6.26),(6.27) begin at the line L for $\varphi = 0$ and terminate at the boundary of the domain Q as $|\varphi| \to {}^1/_2\pi$.

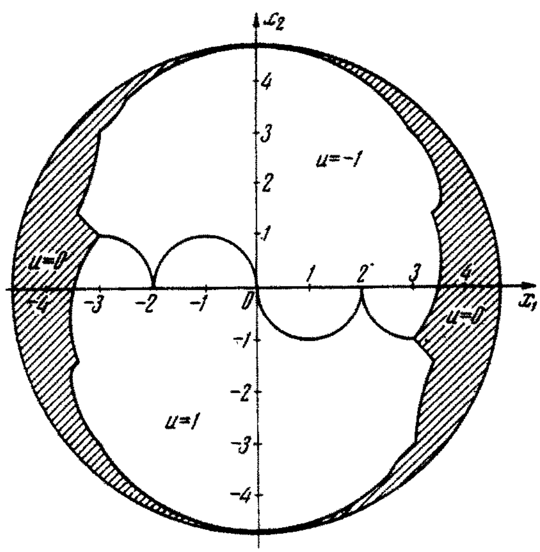

Fig. 5

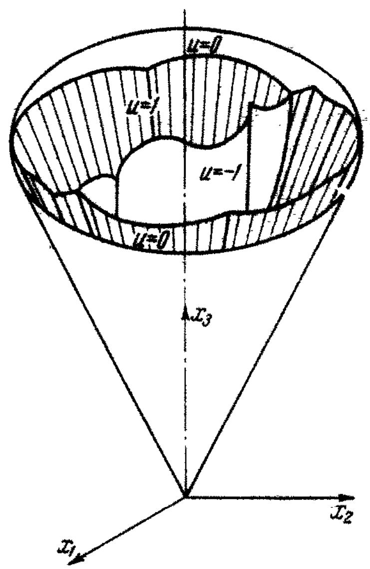

Fig. 6

In Fig. 5 (for $x_3 = {}^3/_2\pi$) the controllability domain Q is split by curves (6.26) − (6.28) into domains where the optimal control at the initial instant is equal to −1, 0 (shaded area), and 1.

Considering relations (6.24), (6.26)−(6.28) in the half-space $x_3 > 0$, we gain a full understanding of the optimal control synthesis pattern. This synthesis pattern appears in Fig. 6 (Fig. 5 shows the projection on the plane x_1, x_2 of the cross section $x_3 = {}^3/_2\pi$).

Optimal control synthesis for system (6.1) in the case $\nu = 0$ differs from the case $\lambda = 0$, $\nu < 0$ by the fact that for $\nu = 0$ the set of points where $u = 0$ is of measure zero in the space X_3.

BIBLIOGRAPHY

1. Singh, R. N. P., Functional analysis approach to optimal control problems with multiple constraints on the controlling function. Internat. J. Control Vol. 9, №1, 1969.
2. Pontriagin, L. S., Boltianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., The Mathematical Theory of Optimal Processes. Moscow, Fizmatgiz, 1961.
3. Bellman, R., Glicksberg, I. and Gross, O., Some Problems of the Mathematical Theory of Control Processes (Russian Translation). Moscow, Izd. Inost. Lit., 1962.
4. Krasovskii, N. N., Theory of Motion Control. Moscow, "Nauka", 1968.
5. Boltianskii, V. G., Mathematical Methods of Optimal Control. Moscow, "Nauka", 1969.
6. Liusternik, L. A. and Sobolev, V. I., Elements of functional Analysis. Moscow, "Nauka", 1965.
7. Beckenbach, E. and Bellman, R., Inequalities. Moscow, "Mir", 1965.
8. Formal'skii, A. M., Controllability domains of systems with restricted control resources. Avtomatika i Telemekhanika №3, 1968.
9. Eaton, H. H., An iterative solution to time optimal control. J. Math. Anal. and Appl. Vol. 5, №2, 1962.
10. Shilov, G. E., Mathematical Analysis. Moscow, Fizmatgiz, 1961.
11. Gabasov R. and Gindes, V. B., On optimal processes in linear systems with two restrictions on the controlling forces. Avtomatika i Telemekhanika Vol. 26, №6, 1965.
12. Kalman, R. E., On the general theory of control systems. In: Proceedings of the First International Congress of the International Automatic Control Federation, Vol. 2, Moscow, Izd. Akad. Nauk SSSR, 1961.
13. Athans, M. and Falb, P., Optimal Control. McGraw-Hill, New York, 1966.
14. Krasovskii, N. N., Lectures on Control Theory, ed. 1. Gor'kii Ural University, Sverdlovsk, 1968.

Translated by A. Y.